Weak Converge to An Operator Fractional Brownian Sheet by Martingale Differences

Hongshuai Dai[1,3], Guangjun Shen § [2] and Liangwen Xia [2]

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Abstract

In this paper, inspired by the fractional Brownian sheet, we first introduce the operator fractional Brownian sheet. Then, based on the martingale differences, we give a weak approximation of the operator fractional Brownian sheet.

Keywords: Fractional Brownian motion, operator fractional Brownian motion, fractional Brownian sheet, operator fractional Brownian sheet, martingale differences.

1. Introduction

Self-similar processes, first studied rigorously by Lamperti [15] under the name "semi-stable", are stochastic processes that are invariant in distribution under suitable scaling of time and space. There has been an extensive literature on self-similar processes. We refer to Vervaat [20] for general properties, to Samorodnitsky and Taqqu [19][Chaps.7 and 8] for studies on Gaussian and stable self-similar processes and random fields.

The fractional Brownian motion (fBm) as a well-known self-similar process has been studied extensively. Many results about weak approximation to fBms have been established recently. See [10, 16] and the references therein. Due to some limitations of fBms, many researchers have studied the generalization of fBms. The fractional Brownian sheet introduced by Kamont [13] is one generalization of fBms. Scholars also have studied the limit theorems for this kind of processes. For more information, refer to [2, 3] and [21, 22].

The definition of self-similarity has been extended to allow scaling by linear operators on multidimensional space $\mathbb{R}^d$, and the corresponding processes are called operator self-similar processes. We refer to [14], [15], [17] and the references therein. We note that Didier and Pipiras [11, 12] introduced the operator fractional Brownian motions (OFBM in short) as an extension of fBms and studied their properties. Similar to fBms, weak limit theorems for OFBMs have also attracted a lot of interest. Recently, Dai and his coauthors [7]-[9] presented some weak limit theorems for some kinds of OFBM processes.

On the other hand, we should point out that, contrast to the extensive study on the extension of fBms, there is little work to study the extension of OFBMs. Inspired by the

[1] School of Statistics, Shandong University of Finance and Economics, Jinan, 250014, China
[2] Department of Mathematics, Anhui Normal University, Wuhu, 241000, China
[3] School of Mathematics, Guangxi University, Nanning 530004, China.
§ Corresponding author.

study of the fractional Brownian sheet, we will introduce a new process, which we call the operator fractional Brownian sheet, and present a weak limit theorem for it.

The rest of the paper is organized as follows. In Section 2, we introduce the operator fractional Brownian sheet and state the main result of this work. The proof of the main result will be detailed in Section 3.

Most of the estimates of this paper contain unspecified constants. An unspecified positive and finite constant will be denoted by C, which may not be the same in each occurrence. Sometimes we shall emphasize the dependence of these constants upon parameters.

2. Preliminaries and Main Results

In this section, we first introduce the operator fractional Brownian sheet. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete and separable probability space and $\{\mathcal{F}_{s,t}; (s,t) \in [0,T] \times [0,S]\}$ be a family of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_{s,t} \subseteq \mathcal{F}_{\hat{s},\hat{t}}$ for any $t < \hat{t}, s < \hat{s}$, where S and T are positive numbers. Moreover, we denote by $\Delta_{t,s} X(t^{'}, s^{'})$ the increment of X over the rectangle $((t,s),(t^{'},s^{'})]$, that is,

$$\Delta_{t,s} X(t^{'}, s^{'}) = X(t^{'}, s^{'}) - X(t, s^{'}) - X(t^{'}, s) + X(t,s),$$

where $(t,s) < (t^{'}, s^{'})$.

Let $\sigma(A)$ be the collection of all eigenvalues of a linear operator A on $\mathbb{R}^d$. Let

$$\lambda_A = \min\{Re\lambda : \lambda \in \sigma(A)\} \quad \text{and} \quad \Lambda_A = \max\{Re\lambda : \lambda \in \sigma(A)\}. \tag{2.1}$$

Moreover, for any $x \in \mathbb{R}^d$, x^T denotes the transpose of x. We first recall the operator fractional Brownian motion of Riemann-Liouville type introduced by Dai [8]. Let D be a linear operator on $\mathbb{R}^d$ with $\frac{1}{2} < \lambda_D, \Lambda_D < 1$. For any $t \in \mathbb{R}_+$, we define the operator fractional Brownian motion of Riemann-Liouville type $\tilde{X} = \{\tilde{X}(t)\}$ by

$$\tilde{X}(t) = \int_0^t (t-u)^{D-I/2} dW(u), \tag{2.2}$$

where $W(u) = \{W^1(u), ..., W^d(u)\}^T$ is a standard d-dimensional Brownian motion.

In order to introduce the operator fractional Brownian sheet, we need to review the relation between the fBm and the fractional Brownian sheet. By Alòs *et al.* [1], the fBm B^H of Hurst parameter $H \in (0,1)$ has the following integral representation with respect to the standard Brownian motion B:

$$B^H(t) = \int_0^t K_H(t,s) dB(s), \qquad t \geq 0,$$

where K_H is the kernel defined on the set $\{0 < s < t\}$ and it is given by

$$K_H(t,s) = C_H (t-s)^{H-\frac{1}{2}} + C_H(\frac{1}{2} - H) \int_s^t (u-s)^{H-\frac{3}{2}} (1 - (\frac{s}{u})^{\frac{1}{2}-H}) du,$$

with the normalizing constant $C_H > 0$ being given by

$$C_H = \left\{ \frac{2\alpha \Gamma(\frac{3}{2} - \alpha)}{\Gamma(\frac{1}{2} + \alpha)\Gamma(2 - 2\alpha)} \right\}^{\frac{1}{2}}.$$

Recall that a fractional Brownian sheet with parameters $\alpha \in (0,1)$ and $\beta \in (0,1)$ admits an integral representation of the following form (Bardina *et al.* [3]):

$$W^{\alpha,\beta}(t,s) = \int_0^t \int_0^s K_\alpha(t,v) K_\beta(s,u) \tilde{B}(dv,du), \quad (t,s) \in [0,T] \times [0,S] \tag{2.3}$$

where $\tilde{B}$ is a standard Brownian sheet and the deterministic kernel K_α and K_β are the same as the K_H with H being replaced by α and β, respectively.

Inspired by the above, we can define the operator fractional Brownian sheet $X = \{X(t,s), (t,s) \in [0,\ T] \times [0,\ S]\}$ as follows.

Definition 2.1 The operator fractional Brownian sheet $X = \{X(t,s), (t,s) \in [0,\ T] \times [0,\ S]\}$ is defined by

$$X(t,s) = \int_0^t \int_0^s (t-u)^{\frac{D}{2}-\frac{I}{4}} (s-v)^{\frac{D}{2}-\frac{I}{4}} B(du,dv), \tag{2.4}$$

where $B(du,dv) = \big(B^1(du,dv), ..., B^d(du,dv)\big)^T$ with B^i being independent copies of $\tilde{B}$, and $\frac{1}{2} < \lambda_D, \Lambda_D < 1$.

It is obvious that the equation (2.4) is well defined. Next, we study some properties of the process X. We first introduce the following notation. Let $\|x\|_2$ denotes the usual Euclidean norm of $x \in \mathbb{R}^d$. For any linear operator A on $\mathbb{R}^d$, let $\|A\| = \max_{\|x\|_2} \|Ax\|_2$ denote the operator norm of A. Hence, we have

Lemma 2.1 *The random field $X = \{X(t,s), (t,\ s) \in [0,\ T] \times [0,\ S]\}$ is an o.s.s Gaussian random field with exponent D. Moreover, X has a version with continuous sample path a.s..*

Proof: We first check the operator self-similarity. For every $c > 0$, we have

$$\begin{aligned} X(ct,cs) &= \int_0^T \int_0^S (ct-u)_+^{\frac{D}{2}-\frac{I}{4}} (cs-v)_+^{\frac{D}{2}-\frac{I}{4}} dB(u,v) \\ &= c^{D-\frac{I}{2}} \int_0^{ct} \int_0^{cs} (t-\frac{u}{c})^{\frac{D}{2}-\frac{I}{4}} (s-\frac{v}{c})^{\frac{D}{2}-\frac{I}{4}} dB(u,v) \\ &= c^D X(t,s), \end{aligned}$$

since

$$B(cu,\ cv) \stackrel{d}{=} c^{\frac{I}{2}} B(u,\ v), \tag{2.5}$$

where $X \stackrel{d}{=} Y$ means the processes X and Y have the same finite dimensional distributions.

Next, we check the sample continuity. In fact, for any $t, s \in [0,\ T] \times [0,\ S]$ with $\|t-s\|_2 < 1$, by some calculations, we have

$$\begin{aligned} &\Delta_s X(t) = \\ &\int_0^T \int_0^S \Big((t_1-u)_+^{\frac{D}{2}-\frac{I}{4}} - (s_1-u)_+^{\frac{D}{2}-\frac{I}{4}}\Big)\Big((t_2-v)_+^{\frac{D}{2}-\frac{I}{4}} - (s_2-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big) B(du,dv) \end{aligned} \tag{2.6}$$

Hence,

$$\big\|\Delta_s X(t)\big\|_2^2 = \sum_{i=1}^{d}\Big(\int_0^T\int_0^S\sum_{j=1}^{d}F_{i,j}(t,s,u,v)B^j(du,dv)\Big)^2, \tag{2.7}$$

where

$$F(t,s,u,v)=\Big((t_1-u)_+^{\frac{D}{2}-\frac{I}{4}}-(s_1-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big)\Big((t_2-v)_+^{\frac{D}{2}-\frac{I}{4}}-(s_2-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big)=\big(F_{i,j}(t,s,u,v)\big)_{d\times d}.$$

Noting $\int_0^T\int_0^S\sum_{j=1}^{d}F_{i,j}(t,s,u,v)B^j(du,dv)$ is a Gaussian random filed, it follows from (2.7) that

$$\mathbb{E}\Big[\big\|\Delta_s X(t)\big\|_2^k\Big]\le C\Big[\int_0^T\int_0^S\big\|F(t,s,u,v)\big\|^2dudv\Big]^{\frac{k}{2}}. \tag{2.8}$$

On the other hand, we have

$$\big\|F(t,s,u,v)\big\|\le C\|F_1(t,s,u,v)\|^2\times\|F_2(t,s,u,v)\|^2, \tag{2.9}$$

where

$$F_1(t,s,u,v)=\big(t_1-u\big)_+^{\frac{D}{2}-\frac{I}{4}}-\big(s_1-u\big)_+^{\frac{D}{2}-\frac{I}{4}},$$

and

$$F_2(t,s,u,v)=\big(t_2-v\big)_+^{\frac{D}{2}-\frac{I}{4}}-\big(s_2-v\big)_+^{\frac{D}{2}-\frac{I}{4}}.$$

Now, we look at

$$\int_0^T\big\|F_1(t,s,u,v)\big\|^2dudv.$$

Using the same method as in Dai, Hu and Lee [8], we have

$$\int_0^T\|(t_1-u)_+^{\frac{D}{2}-\frac{I}{4}}-(s_1-u)_+^{\frac{D}{2}-\frac{I}{4}}\|^2du\le C\big\|\big(t_1-s_1\big)^{\frac{D}{2}-\frac{I}{4}}\|^2(t_1-s_1). \tag{2.10}$$

Similar to (2.10),

$$\int_0^S\big\|F_2(t,s,u,v)\big\|^2dv\le C\big\|\big(t_2-s_2\big)^{\frac{D}{2}-\frac{I}{4}}\|^2(t_2-s_2). \tag{2.11}$$

From Maejima and Mason [18], (2.8), (2.9),(2.10) and (2.11), we can get that

$$\begin{aligned}\mathbb{E}\Big[\big\|\Delta_s X(t)\big\|_2^k\Big]&\le C\Big[(t_1-s_1)^{\lambda_D+\frac{1}{2}-2\delta}\times(t_2-s_2)^{\lambda_D+\frac{1}{2}-2\delta}\Big]^{\frac{k}{2}}\\&\le C\big\|t-s\big\|_2^{(\lambda_D+\frac{1}{2}-2\delta)k}.\end{aligned} \tag{2.12}$$

The sample continuity follows from Garsia [6] and (2.12). □

The aim of this paper is to prove a weak convergence to the operator fractional Brownian sheet via martingale differences. In order to reach it, we first recall some facts about martingale differences. Similar to Wang, Yan and Yu [21], we will use the definitions and notations introduced in the basic work of Cairoli and Walsh [5] on stochastic calculus in the plane. For any $z=(t,s)\in[0,\ T]\times[0,\ S]$, let $\mathcal{F}_z:=\mathcal{F}_{t,T}\bigvee\mathcal{F}_{S,s}$, the $\sigma-$ fields generated by $\mathcal{F}_{t,T}$ and $\mathcal{F}_{S,s}$. Now, we recall the definition of the strong martingale.

Definition 2.2 An integrable process $\tilde{Y} = \{Y(z), z \in [0,\ T] \times [0,\ S]\}$ is called a strong martingale if:

(i) Y is adapted;

(ii) Y vanishes on the axes;

(iii) $\mathbb{E}\big[\Delta_z Y(z')|\mathcal{F}_z\big] = 0$ for any $z \le z'$ with the usual partial order.

Let $\big\{\xi^{(n)} = (\xi^{(n)}_{i,j}, \mathcal{F}^{(n)}_{i,j})\big\}_{n\in\mathbb{N}}$ be a sequence such that for all

$$\mathbb{E}\Big[\xi^{(n)}_{i+1,j+1}|\mathcal{F}^{(n)}_{i,j}\Big] = 0,$$

where $\mathcal{F}^{(n)}_{i,j} = \mathcal{F}^{(n)}_{i,n} \bigvee \mathcal{F}^{(n)}_{n,j}$ with $\mathcal{F}^{(n)}_{i,n}$ and $\mathcal{F}^{(n)}_{n,j}$ being the $\sigma-$ fields generated by $\xi^{(n)}_{i,n}$ and $\xi^{(n)}_{n,j}$, respectively. Then we call $\{\xi^{(n)} = (\xi^{(n)}_{i,j}, \mathcal{F}^{(n)}_{i,j})\}_{n\in\mathbb{N}}$ a martingale differences sequence.

It is well known that if the martingale differences sequence $\{\xi^{(n)}\}$ satisfies the following condition

$$\sum_{i=1}^{\lfloor nt\rfloor}\sum_{i=1}^{\lfloor ns\rfloor} (\xi^{(n)}_{i,j})^2 \to t\cdot s$$

in the sense of L^1, then the sequence

$$\sum_{i=1}^{\lfloor nt\rfloor}\sum_{i=1}^{\lfloor ns\rfloor} \xi^{(n)}_{i,j}$$

converges weakly to the Brownian sheet, as n goes to infinity (see for example, Morkvenas [18].) Recently, Wang, Yan and Yu [21] extended this work to the fractional Brownian sheet. If $\{\xi^{(n)}\}$ is a square integrable martingale differences sequence stratifying the following two conditions

$$\lim_{n\to\infty} n(\xi^{(n)}_{i,j}) = 1, \quad a.s. \tag{2.13}$$

for any $1 \le i, j \le n$, and

$$\max_{1\le i,j\le n} |\xi^{(n)}_{i,j}| \le \frac{C}{n}, a.s. \tag{2.14}$$

for some $C \ge 1$, then they [21] constructed a sequence $\{Z_n\}$ to converge weakly to the fractional Brownian sheet. Inspired by these work, we want to study the weak limit theorem for the operator fractional Brownian sheet we introduced in Definition 2.1.

Define

$$\eta^{(n)}_{i,j} = (\xi^{(n)}_{i,j,1}, ..., \xi^{(n)}_{i,j,d})^T, \tag{2.15}$$

and

$$B_n(t,s) = \sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor} \eta^{(n)}_{i,j}, \tag{2.16}$$

where $\xi_{i,j,k}^{(n)}, k=1,\cdots,d$ are independent copies of $\xi_{i,j}^{(n)}$.

From the above, we obtain that $\{(\eta_{i,j}^{(n)}, \mathcal{F}_{i,j}^{(n)})\}_{n\in\mathbb{N}}$ is still a sequence of square integrable martingale differences on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

For any $n\geq 1$, $(t,s)\in[0,1]\times[0,1]$, define

$$\begin{aligned} X_n(t,s) &= \int_0^t\int_0^s (t-u)_+^{\frac{D}{2}-\frac{I}{4}}(s-v)_+^{\frac{D}{2}-\frac{I}{4}}B_n(du,dv) \\ &= n^2\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor}\eta_{i,j}^{(n)}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}(t-u)_+^{\frac{D}{2}-\frac{I}{4}}(s-v)_+^{\frac{D}{2}-\frac{I}{4}}dudv. \end{aligned} \tag{2.17}$$

The main result of this paper is the following.

Theorem 2.1 *The sequence of processes $\{X_n(t,s),(t,s)\in[0,1]\times[0,1]\}$ given by (2.17), as n tends to infinity, converges weakly to the operator fractional Brownian sheet $\{X(t,s);(t,\ s)\in[0,\ 1]\times[0,\ 1]\}$ given by (2.4).*

3. Proofs

The proofs are based on a series of technical results.

Lemma 3.1 *Let $\{X_n(t,s)\}$ be the family of processes defined by (2.17). Then for any $s<t<u$, there exists a constant C such that*

$$\mathbb{E}\Big[\big\|\Delta_s X_n(t)\big\|_2\big\|\Delta_t X_n(u)\big\|_2\Big]\leq C(u_2-s_2)^H(u_1-s_1)^H, \tag{3.1}$$

where $H=\lambda_D-2\delta+\frac{1}{2}$ with $\delta>0$.

Proof: We choose $(t,s)<(t',s')\in[0,\ 1]\times[0,\ 1]$. Then

$$\begin{aligned} &\Delta_{t,s}X_n(t',s') \\ &= \int_t^{t'}\int_s^{s'}\Big((t'-u)_+^{\frac{D}{2}-\frac{I}{4}}-(t-u)_+^{\frac{D}{2}-\frac{I}{4}}\Big)\Big((s'-v)_+^{\frac{D}{2}-\frac{I}{4}}-(s-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big)B_n(du,dv) \\ &= \sum_{i=1}^{\lfloor nt'\rfloor}\sum_{j=1}^{\lfloor ns'\rfloor}n^2\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\Big((\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\Big) \\ &\quad\cdot\Big((\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big)dudv\eta_{i,j}^{(n)}. \end{aligned}$$

It follows from (2.14) that

$$\mathbb{E}\Big[\big\|\Delta_{t,s}X_n(t^{'},s^{'})\big\|_2^2\Big] = \mathbb{E}\Big[\Big\|\sum_{i=1}^{\lfloor nt'\rfloor}\sum_{j=1}^{\lfloor ns'\rfloor} n^2\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\Big((\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\Big)$$
$$\cdot\Big((\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\Big)dudv\eta_{i,j}^{(n)}\Big\|_2^2\Big]$$
$$\le C\Big(\sum_{i=1}^{\lfloor nt'\rfloor} n\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}}\|(\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\|^2du\Big)$$
$$\cdot\sum_{j=1}^{\lfloor ns'\rfloor} n\Big(\int_{\frac{j-1}{n}}^{\frac{j}{n}}\|(\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\|^2dv\Big)\Big)^2.$$

Using the Cauchy-Schwarz inequality, the above term can be bounded by

$$C\textstyle\sum_{i=1}^{\lfloor nt'\rfloor}\int_{\frac{i-1}{n}}^{\frac{i}{n}}\|(\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\|^2du$$
$$\textstyle\sum_{j=1}^{\lfloor ns'\rfloor}\int_{\frac{j-1}{n}}^{\frac{j}{n}}\|(\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\|^2dv$$
$$\le C\textstyle\int_0^{t'}\|(\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\|^2du$$
$$\textstyle\int_0^{s'}\|(\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\|^2dv$$
$$\le C\textstyle\int_0^{1}\|(\frac{\lfloor nt'\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor nt\rfloor}{n}-u)_+^{\frac{D}{2}-\frac{I}{4}}\|^2du$$
$$\textstyle\int_0^{1}\|(\frac{\lfloor ns'\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}-(\frac{\lfloor ns\rfloor}{n}-v)_+^{\frac{D}{2}-\frac{I}{4}}\|^2dv. \qquad (3.2)$$

From Dai, Hu and Lee [9], we obtain that

$$\int_0^1\|\,(t'-u)_+^{\frac{D}{2}-\frac{I}{4}}-(s'-u)_+^{\frac{D}{2}-\frac{I}{4}}\,\|^2\,du\le C(t'-s')^H.$$

where $H=\lambda_D-2\delta+\frac{1}{2}$. Then (3.2) can be bounded by

$$C\Big(\frac{[nt']-[nt]}{n}\Big)^H\Big(\frac{[ns']-[ns]}{n}\Big)^H. \qquad (3.3)$$

Hence, for any any $s<t<u\in[0,\ 1]\times[0,\ 1]$, we have

$$\mathbb{E}\Big[\big\|\Delta_sX(t)\big\|_2\big\|\Delta_tX(u)\big\|_2\Big]\le C\big[\mathbb{E}\big[\Delta_tX(u)\|_2^2\big]^{\frac{1}{2}}\big[\mathbb{E}\big[\Delta_sX(t)\|_2^2\big]^{\frac{1}{2}}$$
$$\le C\Big(\frac{[nt_1]-[ns_1]}{n}\Big)^{\frac{H}{2}}\Big(\frac{[nt_2]-[ns_2]}{n}\Big)^{\frac{H}{2}}\Big(\frac{[nu_1]-[nt_1]}{n}\Big)^{\frac{H}{2}}\Big(\frac{[nu_2]-[nt_2]}{n}\Big)^{\frac{H}{2}}$$
$$\le C\Big(\frac{[nu_1]-[ns_1]}{n}\Big)^H\Big(\frac{[nu_2]-[ns_2]}{n}\Big)^H. \qquad (3.4)$$

Hence, if $u_2-s_2\ge\frac{1}{n}$, then

$$\Big|\frac{[nu_2]-[ns_2]}{n}\Big|^H\le C|(u_2-s_2)|^H. \qquad (3.5)$$

Conversely, if $u_2 - s_2 < \frac{1}{n}$, then either u_2 and t_2 or t_2 and s_2 belong to a same subinterval $[\frac{m}{n}, \frac{m+1}{n})$ for some integer m. Hence (3.5) still holds. The second term follows a similar discussion. From the above argument, the proof is now completed. □

Using the criterion given by Bickel and Wickel [4], and noting that $X_n(t,s)$ are null on the axes, and by lemma 3.1, we can get the following lemma.

Lemma 3.2 *The sequence* $\{X_n(t,s); (t,s) \in [0,\ 1] \times [0,\ 1]\}$ *is tight.*

Now, in order to prove Theorem 2.1, it suffices to show the following theorem which states that the law of all possible weak limits is the law of a operator fractional Brownian sheet.

Theorem 3.1 *The family of processes* $X_n(t,s)$ *defined by* (2.17) *converges, as* n *tends to infinity, to the operator fractional Brownian sheet* X *in the sense of finite-dimensional distribution.*

In order to prove Theorem 3.1, we need a technical result. Before we present this lemma, we first introduce the following notation.

$$(t-u)_+^{\frac{D}{2}-\frac{I}{4}} = \left(\tilde{K}_{i,j}(t,u)\right)_{d\times d}$$

and

$$\left(\frac{\lfloor nt \rfloor}{n} - u\right)_+^{\frac{D}{2}-\frac{I}{4}} = \left(\tilde{K}^n_{i,j}(t,u)\right)_{d\times d}.$$

Then, we have

Lemma 3.3 *For any* $(t_k, s_k), (t_l, s_l) \in [0,1] \times [0,1]$ *and* $q, m \in \{1, \cdots, d\}$, *we have*

$$\begin{aligned} n^4 \textstyle\sum_{i=1}^n \sum_{j=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}^n_{q,m}(t_k,u)\tilde{K}^n_{m,q}(s_k,v)dudv \\ \textstyle\int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}^n_{q,m}(t_l,u)\tilde{K}^n_{m,q}(s_l,v)dudv(\xi^{(n)}_{i,j,q})^2, \end{aligned} \tag{3.6}$$

converges to

$$\int_0^1 \int_0^1 \tilde{K}_{q,m}(t_k,u)\tilde{K}_{m,q}(s_k,v)\tilde{K}_{q,m}(t_l,u)\tilde{K}_{m,q}(s_l,v)dudv,\ a.s. \tag{3.7}$$

as n *tends to infinity.*

Proof: It is obvious that (3.6) is equivalent to

$$\begin{aligned} n^2 \textstyle\sum_{i=1}^n n \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{K}^n_{q,m}(t_k,u)du \int_{\frac{i-1}{n}}^{\frac{i}{n}} \tilde{K}^n_{q,m}(t_l,u)du \\ \cdot \textstyle\sum_{j=1}^n n \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}^n_{m,q}(s_k,v)dv \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{K}^n_{m,q}(s_l,v)dv(\xi^{(n)}_{i,j,q})^2. \end{aligned} \tag{3.8}$$

Using the same method as the proof of Lemma 8 in Dai, Hu and Lee[9], we can prove the lemma. □

Next, we prove the Theorem 3.1.

Proof of Theorem 3.1. Let $a_1, ..., a_Q \in \mathbb{R}$ and $(t_1, s_1), ..., (t_Q, s_Q) \in [0,1] \times [0,1]$. We see that the random variable

$$Y_n = \sum_{k=1}^{Q} a_k X_n(t_k, s_k)$$

converges in distribution, as n tends to infinity, to the Gaussian random vector

$$\tilde{X} = \sum_{k=1}^{Q} a_k X(t_k, s_k).$$

By the well-known Cramér-wold device, for example see Whitt [23], in order to prove the above statement, we only need to show that as $n \to \infty$

$$bY_n \xrightarrow{D} b\tilde{X}, \tag{3.9}$$

where $b = (b_1, b_2, \cdots, b_d)$ and $\xrightarrow{D}$ denotes convergence in distribution.

For conciseness of the paper, let

$$(t-u)_+^{\frac{D}{2}-\frac{I}{4}}(s-v)_+^{\frac{D}{2}-\frac{I}{4}} = K(t,s,u,v) = \{K_1(t,s,u,v), \cdots, K_d(t,s,u,v)\}^T,$$

where

$$K_j(t,s,u,v) = \big(K_{j,1}(t,s,u,v), \cdots, K_{j,d}(t,s,u,v)\big).$$

Then, we have

$$\begin{aligned} bY_n &= \sum_{q=1}^{d}\sum_{k=1}^{Q}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_q(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor ns_k\rfloor}{n}, u.v)\eta_{i,j}^{(n)} dudv \\ &= \sum_{m=1}^{d}\sum_{q=1}^{d}\sum_{k=1}^{Q}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor ns_k\rfloor}{n}, u, v)\xi_{i,j,m}^{(n)} dudv, \end{aligned}$$

and

$$b\tilde{X} = \sum_{m=1}^{d}\sum_{q=1}^{d}\sum_{k=1}^{Q}\int_0^1\int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv).$$

Since $\xi_{i,j,m}^{(n)}$, $m = 1, \cdots, d$ are independent, in order to prove (3.9), we only need to show

$$\begin{aligned} &\sum_{q=1}^{d}\sum_{k=1}^{Q}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor ns_k\rfloor}{n}, u, v)\xi_{i,j,m}^{(n)} dudv \xrightarrow{D} \\ &\qquad \sum_{q=1}^{d}\sum_{k=1}^{Q}\int_0^1\int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv). \end{aligned} \tag{3.10}$$

For convenience, we introduce the following notation.

$$Y_{i,j}^{(n)} = \sum_{q=1}^{d}\sum_{k=1}^{Q} n^2 \int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} a_k b_q K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor ns_k\rfloor}{n}, u, v)\xi_{i,j,m}^{(n)} dudv.$$

Then, (3.10) can be rewritten as

$$\sum_{i=1}^{\lfloor nt\rfloor}\sum_{j=1}^{\lfloor ns\rfloor} Y_{i,j}^{(n)} \xrightarrow{D} \sum_{q=1}^{d}\sum_{k=1}^{Q}\int_0^1\int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v) B^m(du, dv). \tag{3.11}$$

Inspired by Wang, Yan and Yu [21], in order to prove (3.11), we first prove the following Lindeberge condition

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Big[(Y_{i,j}^{(n)})^2 1_{\{|Y_{i,j}^{(n)}|>\varepsilon\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}\Big] = 0 \tag{3.12}$$

for all $\varepsilon > 0$.

In fact, we have

$$\begin{aligned}
&\Big(n^2\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor st_k\rfloor}{n}, u, v)\xi_{i,j,m}^{(n)} dudv\Big)^2 \\
&\qquad\le n^4\big(\xi_{i,j,m}^{(n)}\big)^2\Big(\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} \big|K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor st_k\rfloor}{n}, u, v)\big| dudv\Big)^2 \\
&\qquad\le Cn^2\big(\xi_{i,j,m}^{(n)}\big)^2\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} \big|K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor st_k\rfloor}{n}, u, v)\big|^2 dudv.
\end{aligned} \tag{3.13}$$

We also note that for some $\delta > 0$ with $\lambda_D - \frac{1}{2} - 2\delta > 0$,

$$\begin{aligned}
\int_{\frac{i-1}{n}}^{\frac{i}{n}} \big\|(t-u)_+^{\frac{D}{2}-\frac{I}{4}}\big\|^2 du &\le C\int_{\frac{i-1}{n}}^{\frac{i}{n}}(1-u)_+^{\lambda_D-\frac{1}{2}-2\delta} du \\
&\le C\int_0^{\frac{1}{n}}(1-u)_+^{\lambda_D-\frac{1}{2}-2\delta} du,
\end{aligned} \tag{3.14}$$

since $(1-u)_+^{\lambda_D-\frac{1}{2}-2\delta}$ is decreasing in u.

Noting the form of K, it follows from (3.13) and (3.14) that

$$\begin{aligned}
&\Big(n^2\int_{\frac{i-1}{n}}^{\frac{i}{n}}\int_{\frac{j-1}{n}}^{\frac{j}{n}} K_{q,m}(\frac{\lfloor nt_k\rfloor}{n}, \frac{\lfloor st_k\rfloor}{n}, u, v)\xi_{i,j,m}^{(n)} dudv\Big)^2 \\
&\qquad\le Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2,
\end{aligned} \tag{3.15}$$

where

$$\delta_n = \int_0^{\frac{1}{n}}(1-u)_+^{\lambda_D-\frac{1}{2}-2\delta} du.$$

It follows from (3.13) and (3.15) that

$$\left(Y_{i,j}^{(n)}\right)^2 \leq C\sum_{q=1}^{d}\sum_{k=1}^{Q} n^2 a_k^2 b_q^2 (\xi_{i,j,m}^{(n)})^2\delta_n^2. \tag{3.16}$$

On the other hand,

$$\{|Y_{i,j}^{(n)}| > \varepsilon\} = \{|Y_{i,j}^{(n)}|^2 > \varepsilon^2\}. \tag{3.17}$$

Hence, from (3.17) and (3.16),

$$\Big\{|Y_{i,j}^{(n)}| > \varepsilon\Big\} \subseteq \Big\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2 > \epsilon^2\Big\}. \tag{3.18}$$

Consequently,

$$\begin{aligned}\mathbb{E}[(Y_{i,j}^{(n)})^2 1_{\{|Y_{i,j}^{(n)}|>\varepsilon\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}] &\leq C\mathbb{E}[n^2(\xi_{i,j,m}^{(n)})^2\delta_n^2 1_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}] \\ &\leq C\delta_n^2\mathbb{E}[1_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}]\end{aligned} \tag{3.19}$$

for all $i, j = 1, 2, ..., n$. Hence, from (2.13) and (3.19),

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[(Y_{i,j}^{(n)})^2 1_{\{|Y_{i,j}^{(n)}|>\varepsilon\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}] \\ &\quad\leq \sum_{i=1}^{n}\sum_{j=1}^{n} C\delta_n^2\mathbb{E}[1_{\{Cn^2(\xi_{i,j,m}^{(n)})^2\delta_n^2>\varepsilon^2\}}\Big|\mathcal{F}_{i-1,j-1}^{(n)}] \\ &\quad\leq C\delta_n^2\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[1_{\{C\delta_n^2>\varepsilon^2\}}] \to 0 \quad (n\to\infty),\end{aligned}$$

because $\delta_n \to 0$ implies $1_{\{C\delta_n^2>\varepsilon^2\}} \to 0$.

In order to prove (3.9), we also need to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big[Y_{i,j}^{(n)}\Big]^2 \xrightarrow{\mathbb{P}} \mathbb{E}\Big[\sum_{q=1}^{d}\sum_{k=1}^{Q}\int_0^1\int_0^1 a_k b_q K_{q,m}(t_k, s_k, u, v)B^m(du, dv)\Big]. \tag{3.20}$$

For convenience, we define

$$\tilde{B}^m(t, s, u, v) = \sum_{q=1}^{d} b_q K_{q,m}(t, s, u, v).$$

Note that the right- hand of (3.20) is equivalent to

$$\sum_{i,j=1}^{Q} a_i a_j\int_0^1\int_0^1 \tilde{B}^m(t_i, s_i, u, v)\tilde{B}^m(t_j, t_j, u, v)dudv. \tag{3.21}$$

Next, we look at the left-side hand of (3.20). In fact, we have

$$Y_{i,j}^{(n)} = \sum_{k=1}^{Q} a_k \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{B}^m(\frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v) \xi_{i,j,m}^{(n)} dudv. \tag{3.22}$$

Hence,

$$\begin{aligned}\left(Y_{i,j}^{(n)}\right)^2 = (\xi_{i,j,m}^{(n)})^2 \sum_{k,l=1}^{Q} a_k a_l \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{B}^m(\frac{\lfloor nt_k \rfloor}{n}, \frac{\lfloor ns_k \rfloor}{n}, u, v) dudv \\ \cdot \int_{\frac{i-1}{n}}^{\frac{i}{n}} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \tilde{B}^m(\frac{\lfloor nt_l \rfloor}{n}, \frac{\lfloor ns_l \rfloor}{n}, u, v) dudv.\end{aligned} \tag{3.23}$$

Here, we also should point out that the entry $K_{q,m}(t,s,u,v)$ takes the form of

$$\sum_{i=1}^{d} \tilde{K}_{q,i}(t,u) \tilde{K}_{q,i}(s,v) \tilde{K}_{i,m}(t,u) \tilde{K}_{i,m}(s,v).$$

Hence, it follows from Lemma 3.3 and (3.21) to (3.23) that (3.20) holds.

From the above arguments, we can easily get that the theorem holds. □

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